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Mean drift forces on arrays of bodies due to incident long waves

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The scattering of long water waves by an array of bodies is investigated using the method of matched asymptotic expansions. Two particular geometries are considered: a group of vertical cylinders extending throughout the depth and a group of floating hemispheres. From these solutions, the low-frequency limit of the ratio of the mean drift force on a group of N bodies to that on a single body is calculated. For a wide range of circumstances this drift-force ratio is N^2, which is in agreement with previous numerical work. Further drift-force enhancement is possible for certain configurations of vertical cylinders.

1. Introduction

Bodies floating in irregular seas may experience low-frequency oscillations as a result of second-order interactions between different frequency components of the incident waves. Newman (1974) has shown that the slowly varying forces that give rise to these oscillations can be approximated in terms of the mean drift forces on the body in regular waves. Though also a second-order effect, the mean drift force is determined by the first-order solution and so is more easily calculated than the slowly varying force. Consequently, there has been considerable interest in the mean drift force on a body. For further information concerning the various components of the second-order force see, for example, Pinkster (1979).

Recent numerical results of Eatock-Taylor & Hung (1985) concerning mean drift forces on multi-column structures have shown that interaction effects between structural elements may be very important at low frequencies. Their work suggests that for certain geometries the mean horizontal drift force on a group of N bodies may be of the order of N^2 times the forces on a single body when the incident waves are long compared with the overall body size. In a discussion accompanying that paper the present author indicated briefly how this 'N^2 law' may be verified analytically for an array of widely spaced cylinders. In the present work a more rigorous and extensive investigation of this behaviour is undertaken. In particular, the long-wave limit of the mean drift force is calculated for an array of vertical cylinders extending throughout the water depth and for an array of hemispheres floating on water of infinite depth. In the former case the N^2 law does not always hold – for closely spaced bodies there may be additional enhancement effects – but for an array of hemispheres the law is exact in the low-frequency limit. Consideration of a known solution for scattering by a single truncated cylinder suggests that any additional enhancement, as exhibited in the vertical-cylinder case, will be small for most floating bodies.

As stated above, the mean drift force is calculated from the solution of the linear diffraction problem. A number of authors have proposed solution procedures for wave scattering by an array of arbitrary bodies that can, in principle, be used for all wave frequencies and body spacings, for example that due to Kagemoto & Yue (1986). However, none of these methods can yield, easily, analytical results for the long-wave limit. In the present work the problem of the scattering of long waves by an array of bodies is solved using the method of matched asymptotic expansions. The basic procedure is similar to that used by Balsa (1977, 1982, 1983) for work in acoustics on low-frequency flow through an array of bodies; consequently some details of the matching will not be given here.

The horizontal mean drift force may be calculated from the linear diffraction potential in two ways, either by direct integration of a quadratic term over the body surface or indirectly from the far-field potential. Here the latter procedure, due to Maruo (1960), is employed, so the leading-order approximation to the far-field potential for long waves is required. The general formulation of the scattering problem is given in §2, while the solutions for an array of cylinders and an array of hemispheres are described in §§3 and 4 respectively. Finally, in §5, expressions for the mean drift force are presented and discussed.

2. General formulation

A plane wavetrain of amplitude A and frequency ω is incident upon an array of N fixed bodies in water of constant depth h. Cartesian coordinates (x, y, z) are defined so that the (x, y)-plane corresponds to the mean free surface and the z-axis is directed vertically downwards. The origin of coordinates is chosen to be within the array. It will also be convenient to employ cylindrical polar coordinates (r, ψ, z) and spherical polar coordinates (ρ, θ, ψ), defined as shown in figure 1. Coordinate systems with origin at the centre of the waterplane area of each body will also be used: a subscript j will indicate coordinates associated with body j. The origin of coordinate system j is at $(x, y, z) = (\xi_j, \eta_j, 0)$, while the position of body l relative to body j has polar coordinates $(r_j, \psi_j) = (R_{jl}, \alpha_{jl})$. A plan view of the array is given in figure 2.

The usual assumptions of the linearized theory of water waves are made, including those of inviscid, irrotational flow. Hence the fluid motion is described by a velocity potential

$$\Phi_{\mathrm{T}}(x, y, z, t) = \mathrm{Re}\left\{-\frac{\mathrm{i}gA}{\omega}\phi_{\mathrm{T}}(x, y, z)\,\mathrm{e}^{-\mathrm{i}\omega t}\right\}, \tag{2.1}$$

where g is the acceleration due to gravity. The complex-valued function $\phi_{\mathrm{T}}(x, y, z)$ satisfies

$$\nabla^2\phi_{\mathrm{T}} = 0 \tag{2.2}$$

within the fluid, the linearized free-surface condition

$$\frac{\partial\phi_{\mathrm{T}}}{\partial z} + \frac{\omega^2}{g}\phi_{\mathrm{T}} = 0, \quad z = 0 \tag{2.3}$$

(excluding the waterplane area of each body) and zero-flux conditions on the bed $z = h$ and the wetted surface of each body. The scattered part of the wave field must also satisfy a radiation condition allowing only outward-propagating waves at large horizontal distances from the array.

The scattering problem is solved by the method of matched asymptotic expansions under the assumptions that $\mu = kl \ll 1$ and $\epsilon = a/l \ll 1$ (which together imply $\mu\epsilon =$

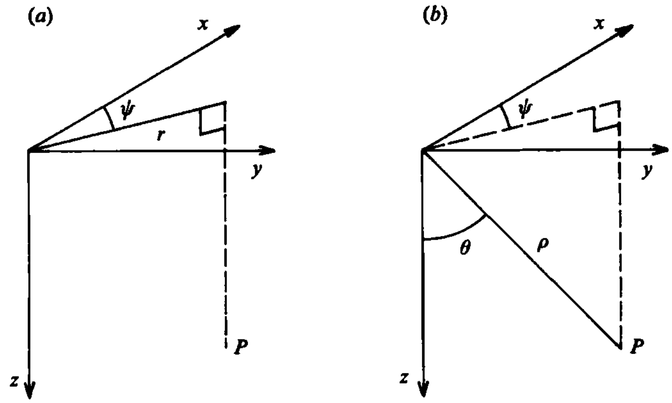

FIGURE 1. Definition of (a) cylindrical polar coordinates (r, ψ, z) and (b) spherical polar coordinates (ρ, θ, ψ) of the field point P.

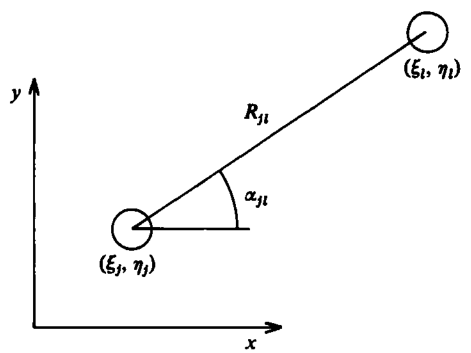

FIGURE 2. Plan view of array showing bodies j and l.

$ka \ll 1$). Here k is the wavenumber, related to the radian frequency ω through the dispersion relation

$$\omega^2 = gk \tanh kh, \tag{2.4}$$

l is a typical body spacing and a is a typical body radius. The above assumptions indicate that the waves are long relative to the body separation and that the bodies are widely spaced relative to body size, though interaction effects beyond the lowest order in ϵ will be included in the following analysis. In addition, the overall size of the array is restricted to be much less than the wavelength, and so the array must be of finite horizontal extent.

Following Balsa (1982, 1983), three flow regions are distinguished. These are: an outer region at large distances from the array where the lengthscale is k^{-1}; an intermediate region within the array (but not 'close' to any body) where the lengthscale is l; and N inner regions surrounding each body where the lengthscale is a. In the outer region the scattered wave appears to be the result of singularities at a single origin, whilst in the intermediate region the disturbance appears to be generated by singularities at the origin of each body coordinate system.

The previous work of Balsa makes it unnecessary to give full details of the solution derivation here. In particular the correct gauge functions for the various asymptotic expansions will be assumed from the outset. For each of the two body geometries considered the solution is presented in the following way. The potentials in each of the three regions are expanded in terms of gauge functions in μ, and the outer and intermediate expansions matched so that the leading-order outer solution is determined in terms of constants in the intermediate solution. The potentials in the intermediate and inner regions are further expanded in terms of gauge functions in ϵ. The body boundary conditions are satisfied in the inner regions, and matching determines the intermediate solution and hence the outer solution. The final form of the far-field potential is a double expansion in μ and ϵ.

3. An array of vertical cylinders

Each body is taken to be a vertical circular cylinder of radius a, extending throughout the depth of the fluid. The incident wave travels in the direction of increasing x so that ϕ_T may be written

$$\phi_{\mathrm{T}}(x,y,z) = \left\{e^{ikx} + \phi(x,y)\right\} \frac{\cosh k(z-h)}{\cosh kh}, \tag{3.1}$$

where the first term within the braces represents the incident wave and ϕ the scattered wave. Substitution of this expression for ϕ_T into (2.2) gives

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0. \tag{3.2}$$

The free-surface condition (2.3) and the zero-flux condition on $z = h$ are satisfied identically because of the special choice of form of ϕ_T. The function ϕ must satisfy the boundary conditions

$$\frac{\partial\phi}{\partial r_j} = -\frac{\partial}{\partial r_j}(e^{ikx}), \quad r_j = a \quad (j = 1, 2, \ldots, N) \tag{3.3}$$

on the wetted surfaces of the cylinders.

3.1. *Outer/intermediate matching*

Scaled coordinates for the outer region are defined by

$$\hat{x} = kx, \quad \hat{y} = ky, \quad \hat{r} = kr \tag{3.4}$$

so that, from (3.2), $\hat{\phi}(\hat{x},\hat{y})$ $(\equiv \phi(x,y))$ satisfies

$$\frac{\partial^2\hat{\phi}}{\partial \hat{x}^2} + \frac{\partial^2\hat{\phi}}{\partial \hat{y}^2} + \hat{\phi} = 0 \tag{3.5}$$

in the fluid region. In the intermediate region appropriate scaled coordinates are defined by

$$\bar{x} = \frac{x}{l}, \quad \bar{y} = \frac{y}{l}, \quad \bar{r} = \frac{r}{l} \tag{3.6}$$

and $\bar{\phi}(\bar{x},\bar{y})$ $(\equiv \phi(x,y))$ satisfies

$$\frac{\partial^2\bar{\phi}}{\partial \bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial \bar{y}^2} + \mu^2\bar{\phi} = 0 \tag{3.7}$$

within the fluid. The outer and intermediate potentials are expanded in μ as

$$\hat{\phi} = \mu^2\hat{\phi}^{(2)} + O(\mu^3) \tag{3.8}$$

and

$$\bar{\phi} = \mu\bar{\phi}^{(1)} + \mu^2 \ln\mu\bar{\phi}^{(2.1)} + \mu^2\bar{\phi}^{(2)} + O(\mu^3) \tag{3.9}$$

respectively; here only those terms required to determine the outer potential to leading order are displayed explicitly. In the current work the order symbol $O(\mu^s)$ is used to denote terms of order $\mu^s(\ln \mu)^t$ for all t greater than or equal to zero. A superscript within parentheses identifies individual terms within an expansion.

The most general solution for $\hat{\phi}^{(2)}$ satisfying (3.5) and the radiation condition is

$$\hat{\phi}^{(2)} = \hat{A}_0^{(2)} H_0(\hat{r}) + \sum_{m=1}^{\infty} (\hat{A}_m^{(2)} \cos m\psi + \hat{B}_m^{(2)} \sin m\psi) H_m(\hat{r}), \tag{3.10}$$

where H_m denotes the Hankel function of the first kind and order m, and $\hat{A}_0^{(2)}, \hat{A}_1^{(2)}, \ldots, \hat{B}_1^{(2)}, \ldots$ are complex constants to be determined from the matching. Substitution of (3.9) into (3.7) shows that each term in the expansion of $\bar{\phi}$, to the order displayed, is a solution of the two-dimensional Laplace equation. The most convenient form for each $\bar{\phi}^{(t)}$ $(t = 1, 2.1, 2)$ is in terms of singularities within each body; thus

$$\bar{\phi}^{(t)} = \bar{A}_0^{(t)} + \sum_{j=1}^{N} \left\{ \bar{B}_{j0}^{(t)} \ln \bar{r}_j + \sum_{m=1}^{\infty} (\bar{A}_{jm}^{(t)} \cos m\psi_j + \bar{B}_{jm}^{(t)} \sin m\psi_j) \bar{r}_j^{-m} \right\}, \tag{3.11}$$

where $\bar{A}_0^{(t)}, \bar{A}_{j1}^{(t)}, \ldots, \bar{B}_{j0}^{(t)}, \ldots$ are complex constants to be determined. The expansions to $O(\mu^2)$ of $\hat{\phi}$ and $\bar{\phi}$ may now be matched following the principles described by Van Dyke (1975). As a result the outer solution may be written in terms of the constants in the intermediate solution as

$$\hat{\phi}^{(2)} = \frac{\bar{A}_0^{(2)}}{\Gamma} H_0(\hat{r}) + \tfrac{1}{2}\pi \mathrm{i} \sum_{j=1}^{N} (\bar{A}_{j1}^{(1)} \cos \psi + \bar{B}_{j1}^{(1)} \sin \psi) H_1(\hat{r}), \tag{3.12}$$

where
$$\Gamma = 1 + \frac{2\mathrm{i}}{\pi}(\gamma - \ln 2) \tag{3.13}$$

and $\gamma = 0.577215\ldots$ is Euler's constant. Most of the constants in the intermediate solution are still undetermined, though the following relations are noted:

$$\left.\begin{aligned} &\bar{A}_0^{(1)} = 0, \quad \bar{B}_{j0}^{(1)} = 0, \quad j = 1, 2, \ldots, N \\ &\bar{A}_0^{(2.1)} = \frac{2\mathrm{i}}{\pi\Gamma} \bar{A}_0^{(2)} = \sum_{j=1}^{N} \bar{B}_{j0}^{(2)}. \end{aligned}\right\} \tag{3.14}$$

3.2. *Intermediate/inner matching*

The scaled coordinates for the inner region of body j are defined by

$$\tilde{x}_j = \frac{x-\xi_j}{a}, \quad \tilde{y}_j = \frac{y-\eta_j}{a}, \quad \tilde{r}_j = (\tilde{x}_j^2 + \tilde{y}_j^2)^{\frac{1}{2}} \tag{3.15}$$

so that, from (3.2), $\tilde{\phi}_j(\tilde{x}_j, \tilde{y}_j)\,(\equiv \phi(x,y))$ satisfies

$$\frac{\partial^2 \tilde{\phi}_j}{\partial \tilde{x}_j^2} + \frac{\partial^2 \tilde{\phi}_j}{\partial \tilde{y}_j^2} + (\mu\epsilon)^2 \tilde{\phi}_j = 0. \tag{3.16}$$

The body boundary condition (3.3) becomes

$$\frac{\partial \tilde{\phi}_j}{\partial \tilde{r}_j} = -\frac{\partial}{\partial \tilde{r}_j}\{\exp(\mathrm{i}\mu(\epsilon\tilde{x}_j + \bar{\xi}_j))\}, \quad \tilde{r}_j = 1, \tag{3.17}$$

where $\bar{\xi}_j = \xi_j / l$.

As for the intermediate solution, (3.9), the inner solution is posed first as an expansion in μ of the form

$$\tilde{\phi}_j = \mu\tilde{\phi}_j^{(1)} + \mu^2 \ln\mu\tilde{\phi}_j^{(2.1)} + \mu^2\tilde{\phi}_j^{(2)} + O(\mu^3), \tag{3.18}$$

where each term displayed is a solution of Laplace's equation. The individual terms of the expansions in μ for $\bar{\phi}$ and $\tilde{\phi}_j$ are now expanded in ϵ. Beginning with the first non-zero term, the expansions for the inner potential are

$$\tilde{\phi}_j^{(1)} = \epsilon\tilde{\phi}_j^{(1,1)} + \epsilon^2\tilde{\phi}_j^{(1,2)} + \ldots \tag{3.19}$$

and

$$\tilde{\phi}_j^{(2)} = \epsilon\tilde{\phi}_j^{(2,1)} + \epsilon^2 \ln\epsilon\tilde{\phi}_j^{(2,2.1)} + \epsilon^2\tilde{\phi}_j^{(2,2)} + \ldots. \tag{3.20}$$

The appearance of the $\epsilon^2 \ln\epsilon$-term in (3.20) is due to a source-like term in $\tilde{\phi}_j^{(2,2)}$; all other gauge functions are powers of ϵ. Similar expansions are adopted for the intermediate solution, though these contain only integer powers of ϵ, beginning with ϵ^2. The expansion of the intermediate solution in ϵ is equivalent to expanding the constant coefficients in (3.11), and this interpretation will be used here. An expansion for $\tilde{\phi}_j^{(2.1)}$ is not given above as it is particularly simple; it turns out that only constant terms may be matched with the intermediate solution. The corresponding term in the intermediate expansion reduces to the same constant so that

$$\tilde{\phi}_j^{(2.1)} = \bar{\phi}^{(2.1)} = \bar{A}_0^{(2.1)} \tag{3.21}$$

which, by (3.14), is determined by $\bar{A}_0^{(2)}$.

The body boundary conditions for $\tilde{\phi}_j^{(1,1)}$, $\tilde{\phi}_j^{(2,1)}$ and $\tilde{\phi}_j^{(2,2)}$ are, from (3.17),

$$\frac{\partial\tilde{\phi}_j^{(1,1)}}{\partial\tilde{r}_j} = -\mathrm{i}\cos\psi_j, \quad \tilde{r}_j = 1, \tag{3.22}$$

$$\frac{\partial\tilde{\phi}_j^{(2,1)}}{\partial\tilde{r}_j} = \xi_j\cos\psi_j, \quad \tilde{r}_j = 1 \tag{3.23}$$

and

$$\frac{\partial\tilde{\phi}_j^{(2,2)}}{\partial\tilde{r}_j} = \cos^2\psi_j, \quad \tilde{r}_j = 1. \tag{3.24}$$

All other terms on the right-hand side of (3.19) and (3.20) must satisfy the condition of zero-normal derivative on the body surfaces. The general form of $\tilde{\phi}_j^{(s,t)}$, which satisfies Laplace's equation and has zero-normal derivative on body j, is

$$\tilde{\phi}_{j,c}^{(s,t)} = \tilde{A}_{j0}^{(s,t)} + \sum_{m=1}^{\infty} (\tilde{A}_{jm}^{(s,t)} \cos m\psi_j + \tilde{B}_{jm}^{(s,t)} \sin m\psi_j)(\tilde{r}_j^m + \tilde{r}_j^{-m}). \tag{3.25}$$

For $\tilde{\phi}_j^{(1,1)}$, $\tilde{\phi}_j^{(2,1)}$ and $\tilde{\phi}_j^{(2,2)}$ particular solutions satisfying the boundary conditions (3.22)–(3.24) are to be added to this; these are

$$\tilde{\phi}_{j,p}^{(1,1)} = \mathrm{i}\frac{\cos\psi_j}{\tilde{r}_j}, \tag{3.26}$$

$$\tilde{\phi}_{j,p}^{(2,1)} = -\xi_j\frac{\cos\psi_j}{\tilde{r}_j} \tag{3.27}$$

and

$$\tilde{\phi}_{j,p}^{(2,2)} = \tfrac{1}{2}\ln\tilde{r}_j - \frac{1}{4}\frac{\cos 2\psi_j}{\tilde{r}_j^2}. \tag{3.28}$$

Matching these forms of the inner solution with the intermediate solution (3.11) yields

$$\left.\begin{aligned} \bar{A}_{j1}^{(1)} &= \epsilon^2 \mathrm{i}\left\{1-\epsilon^2 \sum_{\substack{p=1\\p\neq j}}^{N} \frac{\cos 2\alpha_{jp}}{\bar{R}_{jp}^2}\right\},\\ \bar{B}_{j1}^{(1)} &= -\epsilon^4 \mathrm{i} \sum_{\substack{p=1\\p\neq j}}^{N} \frac{\sin 2\alpha_{jp}}{\bar{R}_{jp}^2},\\ \bar{A}_0^{(2)} &= \epsilon^2 \frac{\pi \Gamma N}{4\mathrm{i}}, \end{aligned}\right\} \tag{3.29}$$

correct to order ϵ^4. A careful examination of the matching equations shows the expression for $\bar{A}_0^{(2)}$ to be exact. The coefficients in (3.29) determine the outer solution to leading order in μ. Details of the inner and intermediate solutions are not required in the present work, though for completeness they are recorded in Appendix A.

4. An array of floating hemispheres

Let each body be a floating hemisphere (i.e. with the plane face at the mean free surface) of radius a. The water depth is taken to be infinite, that is $kh \to \infty$ in (2.4), so that a wavetrain incident from large negative x is described by the potential

$$\phi_{\mathrm{inc}} = \exp\{K(\mathrm{i}x - z)\}, \tag{4.1}$$

where the wavenumber $K = \omega^2/g$. The scattered wave potential $\phi(x, y, z)$ must satisfy (2.2) and (2.3) with the body boundary conditions

$$\frac{\partial \phi}{\partial \rho_j} = -\frac{\partial}{\partial \rho_j} \mathrm{e}^{K(\mathrm{i}x-z)}, \quad \rho_j = a \quad (j = 1, 2, \ldots, N), \tag{4.2}$$

where ρ_j is a spherical polar coordinate as defined in §2.

4.1. *Outer/intermediate matching*

Scaled coordinates for the outer region are defined by

$$\hat{x} = KX, \quad \hat{y} = Ky, \quad \hat{z} = Kz \tag{4.3}$$

so that $\hat{\phi}(\hat{x}, \hat{y}, \hat{z})\ (\equiv \phi(x, y, z))$ satisfies Laplace's equation within the fluid region and the free-surface condition

$$\frac{\partial \hat{\phi}}{\partial \hat{z}} + \hat{\phi} = 0, \quad \hat{z} = 0. \tag{4.4}$$

For the intermediate region appropriate scaled coordinates are

$$\bar{x} = \frac{x}{l}, \quad \bar{y} = \frac{y}{l}, \quad \bar{z} = \frac{z}{l} \tag{4.5}$$

so that $\bar{\phi}(\bar{x}, \bar{y}, \bar{z})\ (\equiv \phi(x, y, z))$ satisfies Laplace's equation in the fluid region and the free-surface condition

$$\frac{\partial \bar{\phi}}{\partial \bar{z}} + \mu \bar{\phi} = 0, \quad \bar{z} = 0, \tag{4.6}$$

where $\mu = Kl$. In contrast to the vertical cylinder case, the leading-order outer solution is fully determined by the leading-order intermediate solution; put

$$\hat{\phi} = \mu^2 \hat{\phi}^{(2)} + O(\mu^3) \tag{4.7}$$

and

$$\bar{\phi} = \mu \bar{\phi}^{(1)} + O(\mu^2). \tag{4.8}$$

The outer solution $\hat{\phi}^{(2)}$ may be expressed in terms of generalized wave sources (Thorne 1953) so that

$$\hat{\phi}^{(2)} = \hat{A}_0^{(2)} \Psi_0(\hat{r}, \hat{z}) + \sum_{m=1}^{\infty} (\hat{A}_m^{(2)} \cos m\psi + \hat{B}_m^{(2)} \sin m\psi)\, \Psi_m(\hat{r}, \hat{z}), \tag{4.9}$$

where

$$\Psi_m(\hat{r}, \hat{z}) = \int_0^\infty \frac{k^{m+1}}{k-1} \mathrm{e}^{-k\hat{z}} J_m(k\hat{r})\, \mathrm{d}k, \tag{4.10}$$

and J_m is a Bessel function of the first kind and order m. Each source individually satisfies Laplace's equation, the free-surface condition (4.4) and the radiation condition. The intermediate solution $\bar{\phi}^{(1)}$ is constructed from singularities within each body that satisfy Laplace's equation and, from (4.6) and (4.8),

$$\frac{\partial \bar{\phi}^{(1)}}{\partial \theta_j} = 0, \quad \theta_j = \tfrac{1}{2}\pi; \tag{4.11}$$

thus

$$\begin{aligned}\bar{\phi}^{(1)} = \sum_{j=1}^{N} \Bigg\{ & \sum_{n=0}^{\infty} \bar{A}_{j,0n}^{(1)} \frac{P_{2n}(\cos\theta_j)}{\bar{\rho}_j^{2n+1}} \\ & + \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} (\bar{A}_{j,mn}^{(1)} \cos 2m\psi_j + \bar{B}_{j,mn}^{(1)} \sin 2m\psi_j) \frac{P_{2n}^{2m}(\cos\theta_j)}{\bar{\rho}^{2n+1}} \\ & + \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (\bar{C}_{j,mn}^{(1)} \cos(2m+1)\psi_j + \bar{D}_{j,mn}^{(1)} \sin(2m+1)\psi_j) \frac{P_{2n+1}^{2m+1}(\cos\theta_j)}{\bar{\rho}^{2n+2}} \Bigg\},\end{aligned} \tag{4.12}$$

where P_n^m is the associated Legendre function of degree n and order m (note that (4.11) is satisfied by taking $m+n$ to be even). The complex constants $\hat{A}_0^{(2)}, \hat{A}_1^{(2)}, \ldots,$ $\hat{B}_1^{(2)}, \ldots$ and $A_{j,00}^{(1)}, \ldots$ are determined by the matching. The expansion of Ψ_m in spherical polar coordinates, required for the matching, is given by Hulme (1982), along with a number of other results relevant to the present work. The leading-order outer solution is

$$\hat{\phi}^{(2)} = \Psi_0(\hat{r}, \hat{z}) \sum_{j=1}^{N} \bar{A}_{j,00}, \tag{4.13}$$

which depends only on the magnitude of the apparent source within each body.

4.2. *Intermediate/inner matching*

The scaled coordinates for the inner region of body j are defined by

$$\tilde{x}_j = \frac{x-\xi_j}{a}, \quad \tilde{y}_j = \frac{y-\eta_j}{a}, \quad \tilde{z} = \frac{z}{a}, \quad \tilde{\rho}_j = (\tilde{x}_j^2 + \tilde{y}_j^2 + \tilde{z}^2)^{\frac{1}{2}} \tag{4.14}$$

so that $\tilde{\phi}_j(\tilde{x}_j, \tilde{y}_j, \tilde{z})\ (\equiv \phi(x,y,z))$ satisfies Laplace's equation, the free-surface condition

$$\frac{\partial \tilde{\phi}_j}{\partial \tilde{z}} + \mu\epsilon \tilde{\phi}_j = 0, \quad \tilde{z} = 0, \tag{4.15}$$

and the body boundary condition

$$\frac{\partial \tilde{\phi}_j}{\partial \tilde{\rho}_j} = -\frac{\partial}{\partial \tilde{\rho}_j}\{\exp(\mu(\epsilon \mathrm{i}\tilde{x}_j + \mathrm{i}\bar{\xi}_j - \epsilon \tilde{z}))\}, \quad \tilde{\rho}_j = 1. \tag{4.16}$$

The inner solution is expanded in μ as

$$\tilde{\phi}_j = \mu \tilde{\phi}_j^{(1)} + O(\mu^2) \tag{4.17}$$

and then $\tilde{\phi}_j^{(1)}$ is further expanded in ϵ as

$$\tilde{\phi}_j^{(1)} = \epsilon \tilde{\phi}_j^{(1,1)} + \epsilon^2 \tilde{\phi}_j^{(1,2)} + \ldots, \tag{4.18}$$

all the gauge functions being integer powers of ϵ. A similar expansion is adopted for $\bar{\phi}^{(1)}$ though, as previously, it is perhaps simpler to think of it as an expansion of the coefficients in (4.12).

The body boundary condition for $\tilde{\phi}_j^{(1,1)}$ is, from (4.16),

$$\frac{\partial \tilde{\phi}_j^{(1,1)}}{\partial \tilde{\rho}_j} = -(\mathrm{i}\sin\theta_j \cos\psi_j - \cos\theta_j), \quad \tilde{\rho}_j = 1, \tag{4.19}$$

while the remaining terms on the right-hand side of (4.18) have zero-normal derivative on the body surface. The free-surface condition (4.15) yields

$$\frac{\partial \tilde{\phi}_j^{(1,t)}}{\partial \theta_j} = 0, \quad \theta_j = \tfrac{1}{2}\pi \quad (t = 1, 2, \ldots). \tag{4.20}$$

The general solution of Laplace's equation satisfying (4.20) and having zero-normal derivative on $\tilde{\rho}_j = 1$ is

$$\begin{aligned}\tilde{\phi}_{j,c}^{(1,t)} = {} & \sum_{n=0}^{\infty} \tilde{A}_{j,0n}^{(1,t)} \left(\tilde{\rho}_j^{2n} + \frac{2n}{(2n+1)\tilde{\rho}_j^{2n+1}}\right) P_{2n}(\cos\theta_j) \\ & + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} (\tilde{A}_{j,mn}^{(1,t)} \cos 2m\psi_j + \tilde{B}_{j,mn}^{(1,t)} \sin 2m\psi_j)\left(\tilde{\rho}_j^{2n} + \frac{2n}{(2n+1)\tilde{\rho}_j^{2n+1}}\right) P_{2n}^{2m}(\cos\theta_j) \\ & + \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} (\tilde{C}_{j,mn}^{(1,t)} \cos(2m+1)\psi_j + \tilde{D}_{j,mn}^{(1,t)} \sin(2m+1)\psi_j) \\ & \qquad \times \left(\tilde{\rho}_j^{2n+1} + \frac{2n+1}{(2n+2)\tilde{\rho}_j^{2n+2}}\right) P_{2n+1}^{2m+1}(\cos\theta_j),\end{aligned} \tag{4.21}$$

with the complex constants $\tilde{A}_{j,00}^{(1,t)}, \ldots$ to be determined from the matching. For $\tilde{\phi}_j^{(1,1)}$ there is an additional particular solution, satisfying (4.19),

$$\tilde{\phi}_{j,p}^{(1,1)} = \frac{\mathrm{i}\sin\theta_j \cos\psi_j}{2\tilde{\rho}_j^2} - \sum_{n=0}^{\infty} \frac{E_n P_{2n}(\cos\theta_j)}{(2n+1)\tilde{\rho}_j^{2n+1}},$$

where, from Hulme (1982, equation B10),

$$\frac{E_n}{4n+1} = \int_0^1 P_1(\mu) P_{2n}(\mu)\,\mathrm{d}\mu = \frac{(-1)^{n+1}(2n)!}{2^{2n+1}(n!)^2(2n-1)(n+1)}. \tag{4.23}$$

Matching with the intermediate solution gives

$$\bar{A}_{j,00} = \tfrac{1}{2}\epsilon^2, \quad j = 1, 2, \ldots, N, \tag{4.24}$$

exactly, which determines the outer solution to leading order in μ. Further details of the intermediate and inner solutions are given in Appendix A.

5. The mean drift force

The mean drift forces and moments on a body have been related to the far field of the first-order scattering potential by Maruo (1960) and Newman (1967). The result for the mean horizontal drift force in the direction of wave advance is

$$f_x^{(2)} = \frac{\rho g A^2}{2\pi k}\left(1+\frac{2kh}{\sinh 2kh}\right)\int_0^{2\pi}(1-\cos\psi)\,|\mathscr{A}(\psi)|^2\,\mathrm{d}\psi \tag{5.1}$$

where ρ is here the fluid density and $\mathscr{A}(\psi)$ is related to the far-field form of the scattered wave through

$$\phi \sim \frac{\cosh k(z-h)}{\cosh kh}\mathscr{A}(\psi)\left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}kr-\mathrm{i}\pi/4}, \quad kr\to\infty. \tag{5.2}$$

The far-field potential in (3.12) for the cylinder array may be written in the form of (5.2) using the expansions of the Hankel functions for large argument (Abramowitz & Stegun 1965, p. 364). The leading-order approximation to the mean horizontal drift force on a single cylinder is

$$f_x^{(2)} \sim \tfrac{5}{16}\rho g A^2 a\pi^2\left(1+\frac{2kh}{\sinh 2kh}\right)(\mu\epsilon)^3. \tag{5.3}$$

To demonstrate the effect of interactions within an array, the ratio $F_x^{(2)}$ of the drift force on N bodies to that on an isolated body is calculated. For the cylinder array

$$\lim_{\mu\to 0} F_x^{(2)} = N^2 - \tfrac{6}{5}\epsilon^2 N\sum_{j=1}^{N}\sum_{\substack{p=1\\p\neq j}}^{N}\frac{\cos 2\alpha_{jp}}{\bar{R}_{jp}^2}+O(\epsilon^4), \tag{5.4}$$

which confirms the N^2 enhancement found previously and also reveals higher-order interaction effects. The presence of these additional terms shows that the geometry of an array may affect wave forces even for very long waves. The existence of the higher-order terms is consistent with the calculations for two cylinders made by Eatock-Taylor & Hung (1985); for two closely spaced cylinders with the line of centres perpendicular to the direction of wave advance their results show further drift-force enhancement over and above an N^2-fold increase. It is interesting to note that the $O(\epsilon^2)$-term in (5.4) is identically zero for a number of specific geometries. This includes two cylinders with their line of centres at an angle $\frac{1}{4}\pi$ to the direction of wave advance and any number ($\geqslant 3$) of cylinders at the vertices of a regular polygon. For those geometries where the term is not identically zero its sign will change with the direction of wave incidence. A simple calculation shows that, for two cylinders, the enhancement of the drift force is greatest when their line of centres is perpendicular to the wave direction and least when they are aligned with the waves.

The far-field form of the potential for scattering by an array of floating hemispheres is found from (4.13) and (4.24) using the result

$$\Psi_0 \sim \left(\frac{2\pi}{Kr}\right)^{\frac{1}{2}}\exp\{-Kz+\mathrm{i}(Kr+\tfrac{1}{4}\pi)\}, \quad Kr\to\infty \tag{5.5}$$

(Hulme 1982, equation (2.8)). Hence the drift force on a single hemisphere is

$$f_x^{(2)} \sim \tfrac{1}{4}\rho g A^2 a\pi^2(\mu\epsilon)^3, \tag{5.6}$$

to leading order, and for an array of N hemispheres

$$\lim_{\mu \to 0} F_x^{(2)} = N^2. \tag{5.7}$$

In this case the long-wave limit of the drift-force ratio has no higher-order correction term in ϵ. For the cylinder array these terms arise from the dipole-like terms in the far-field potential, but there is no such term in the leading-order approximation to the far field for an array of hemispheres.

The dissimilarity in the far-field potentials for the two cases is a consequence of the difference in the scattering properties of the individual bodies. The vertical cylinder extends throughout the depth whereas the hemisphere is localized near the free surface in deep water. The transition from one extreme to the other may be illustrated by the solution of Miles & Gilbert (1968) for a single truncated (i.e. not extending over the full depth) cylinder in finite-depth water. Using a variational technique they derive an approximation to the far-field potential of the scattered wave in the form

$$\phi(r, \psi, z) \sim \cosh k(z-h) \sum_{m=0}^{\infty} \chi_m H_m(kr) \cos m\psi, \tag{5.8}$$

where

$$\chi_m \propto \frac{G_m - J'_m(ka)}{H'_m(ka)} \tag{5.9}$$

$$G_m = \sqrt{2}\frac{d}{h}\left(1+\frac{\sinh 2kd}{2kd} - 2\delta_{m0}\frac{\sinh^2 kd}{(kd)^2}\right)\sinh kh\left(1+\frac{\sinh 2kh}{2kh}\right)^{-\frac{3}{2}}, \tag{5.10}$$

and d is the clearance beneath the cylinder. Interest is in waves that are long compared with the body radius a so, if kh is taken to be $O(1)$, the size of χ_m depends principally on the relative magnitudes of d/h and $J'_m(ka)$. For $ka \ll 1$,

$$J'_m(ka) \sim \begin{cases} -\frac{1}{2}ka, & m = 0, \\ \dfrac{(\frac{1}{2}ka)^{m-1}}{2(m-1)!}, & m \geqslant 1. \end{cases} \tag{5.11}$$

Hence, if d/h is $O(ka)$, $\chi_0 \sim \chi_1$ while $\chi_m \ll \chi_1$ ($m \geqslant 2$), so that to leading order in ka the far field is of the same form as (3.12) for cylinders extending throughout the depth. Hence it is anticipated that the drift-force ratio will be given approximately by (5.4) for an array of truncated cylinders with little bottom clearance. On the other hand, if $d/h = O(1)$ then $\chi_0 \gg \chi_m$ ($m \geqslant 1$) and the low-frequency limit of the drift-force ratio is well approximated by (5.7).

In the present work attention has been focused on the horizontal drift force on arrays of bodies due to very long waves. For practical purposes it is important to know how the strong interaction effects persist for shorter waves. Though, in principle, the present work could be extended to higher order for this purpose it would be more useful to consider more realistic geometries than those treated here. Eatock-Taylor & Hung (1985, 1986) and Kagemoto & Yue (1986) have reported a number of calculations for various geometries which illustrate the possibility of significant drift-force enhancement over a range of wavelengths. However, more work for a greater variety of geometries is required to obtain a full understanding.

The solutions presented here may be used to calculate other hydrodynamic quantities in addition to the horizontal drift force, and some results are given in Appendix B. Eatock-Taylor & Hung (1986) have already pointed out that radiation damping coefficients will also obey an N^2 enhancement law for low frequencies, and this is confirmed by (B 10). Other quantities such as first-order exciting forces and vertical mean drift forces have a simple N-dependence for arrays of bodies. The common factor linking damping coefficients and the horizontal drift force is that each is related to the wave-making properties of a body, as can be seen from well-known expressions in terms of the far-field radiated or scattered waves (e.g. (5.1)). The other hydrodynamic forces mentioned above are essentially near-field quantities.

One feature of the horizontal-drift-force ratio for an array of cylinders, (5.4), is its dependence on the array configuration. The results given in Appendix B show that similar behaviour is found for other quantities in both the cylinder and hemisphere cases. In particular this is true for the first-order exciting force. It is quite common to assume that the ratio of the exciting force on a body in an array to that on the same body in isolation tends to one in the long-wave limit. The present results show that this need not be the case.

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Appendix A. Intermediate and inner solutions

All terms in the intermediate solutions are given correct to $O(\epsilon^4)$, while terms in the inner solutions are correct to $O(\epsilon^3)$.

A.1. *Vertical cylinders*

Intermediate solution, (3.9):

$$\bar{\phi}^{(1)} = \epsilon^2 \mathrm{i} \sum_{n=1}^{N} \frac{\cos \psi_n}{\bar{r}_n} - \epsilon^4 \mathrm{i} \sum_{n=1}^{N} \sum_{\substack{p=1 \\ p \neq n}}^{N} \frac{\cos(\psi_n - 2\alpha_{np})}{\bar{r}_n \bar{R}^2_{np}}, \tag{A 1}$$

$$\bar{\phi}^{(2.1)} = \tfrac{1}{2}\epsilon^2 N, \quad \text{exactly}; \tag{A 2}$$

$$\begin{aligned}\bar{\phi}^{(2)} = \epsilon^2 &\left\{ \frac{\pi \Gamma N}{4\mathrm{i}} + \frac{1}{2} \sum_{n=1}^{N} \ln \bar{r}_n - \sum_{n=1}^{N} \bar{\xi}_n \frac{\cos \psi_n}{\bar{r}_n} \right\} \\ &+ \epsilon^4 \left\{ \frac{1}{2} \sum_{n=1}^{N} \sum_{\substack{p=1 \\ p \neq n}}^{N} \frac{\cos(\psi_n + \alpha_{np})}{\bar{r}_n \bar{R}_{np}} - \frac{1}{4} \sum_{n=1}^{N} \frac{\cos 2\psi_n}{\bar{r}^2_n} + \sum_{n=1}^{N} \sum_{\substack{p=1 \\ p \neq n}}^{N} \bar{\xi}_n \frac{\cos(\psi_n - 2\alpha_{np})}{\bar{r}_n \bar{R}^2_{np}} \right\}. \end{aligned} \tag{A 3}$$

Inner solution, (3.18):

$$\tilde{\phi}^{(1)}_j = \epsilon \mathrm{i} \frac{\cos \psi_j}{\tilde{r}_j} + \epsilon^2 \mathrm{i} \sum_{\substack{n=1 \\ n \neq j}}^{N} \frac{\cos \alpha_{jn}}{\bar{R}_{jn}} - \epsilon^3 \mathrm{i} \sum_{\substack{n=1 \\ n \neq j}}^{N} \frac{\cos(\psi_j - 2\alpha_{jn})}{\bar{R}^2_{jn}} (\tilde{r}_j + \tilde{r}_j^{-1}); \tag{A 4}$$

$$\tilde{\phi}^{(2.1)}_j = \tfrac{1}{2}\epsilon^2 N, \quad \text{exactly}; \tag{A 5}$$

$$\begin{aligned}\tilde{\phi}^{(2)}_j = -\epsilon \bar{\xi}_j \frac{\cos \psi_j}{\tilde{r}_j} + \tfrac{1}{2}\epsilon^2 \ln \epsilon + \epsilon^2 &\left\{ \frac{\pi \Gamma N}{4\mathrm{i}} + \tfrac{1}{2} \ln \tilde{r}_j - \frac{1}{4} \frac{\cos 2\psi_j}{\tilde{r}^2_j} + \sum_{\substack{n=1 \\ n \neq j}}^{N} \left[\tfrac{1}{2} \ln \bar{R}_{jn} - \bar{\xi}_n \frac{\cos \alpha_{jn}}{\bar{R}_{jn}} \right] \right\} \\ &+ \epsilon^3 \left\{ \sum_{\substack{n=1 \\ n \neq j}}^{N} \left[\frac{1}{2} \frac{\cos(\psi_j - \alpha_{jn})}{\bar{R}_{jn}} + \bar{\xi}_n \frac{\cos(\psi_j - 2\alpha_{jn})}{\bar{R}^2_{jn}} \right] (\tilde{r}_j + \tilde{r}_j^{-1}) \right\}. \end{aligned} \tag{A 6}$$

A.2. *Hemispheres*

Intermediate solution, (4.8):

$$\bar{\phi}^{(1)} = -\tfrac{1}{2}\epsilon^2 \sum_{j=1}^{N} \frac{1}{\bar{\rho}_j} + \epsilon^3 \tfrac{1}{2}\mathrm{i} \sum_{j=1}^{N} \frac{\sin\theta_j \cos\psi_j}{\bar{\rho}_j^2} - \tfrac{5}{24}\epsilon^4 \sum_{j=1}^{N} \frac{P_2(\cos\theta_j)}{\bar{\rho}_j^3}. \tag{A 7}$$

Inner solution, (4.17):

$$\tilde{\phi}_j^{(1)} = \epsilon\left\{\mathrm{i}\frac{\sin\theta_j\cos\psi_j}{2\tilde{\rho}_j^2} - \sum_{n=0}^{\infty} \frac{E_n P_{2n}(\cos\theta_j)}{(2n+1)\,\tilde{\rho}_j^{2n+1}}\right\}$$
$$-\tfrac{1}{2}\epsilon^2 \sum_{\substack{n=1\\ n\neq j}}^{N} \frac{1}{\bar{R}_{jn}} - \epsilon^3\left\{\sum_{\substack{n=1\\ n\neq j}}^{N}\left[\tfrac{1}{2}\mathrm{i}\frac{\cos\alpha_{jn}}{\bar{R}_{jn}^2} + \frac{1}{2}\frac{\cos(\psi_j-\alpha_{jn})}{\bar{R}_{jn}^2}\left(\tilde{\rho}_j + \frac{1}{2\tilde{\rho}_j^2}\right)\sin\theta_j\right]\right\}. \tag{A 8}$$

Appendix B. Other hydrodynamic quantities

Given below are a number of results that may be deduced from the solutions presented. In each case the leading-order behaviour for long waves is given, with higher-order interaction effects included to the order permitted by the inner solutions of Appendix A.

B.1. *Vertical mean drift force*

Calculated by integration over the body surface. The force on a hemisphere within an array is

$$f_z^{(2)} \sim \tfrac{1}{2}\rho g A^2 a\pi\mu\epsilon\left(\tfrac{121}{96} + K + O(\epsilon^3)\right), \tag{B 1}$$

where
$$K = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E_m E_n J_{mn}}{(2m+1)(2n+1)} = 0.02878\ldots. \tag{B 2}$$

E_n is given by (4.23) and

$$J_{mn} = \int_0^1 P_{2m}^1(\mu)\,P_{2n}^1(\mu)\,\mu\,\mathrm{d}\mu. \tag{B 3}$$

Using a standard relation (Abramowitz & Stegun 1965, equation 8.5.4) and integration by parts, it may be shown that

$$2(m+n+1)J_{mn} = (2m+1)\int_0^1 P_{2n}^1(\mu)\,P_{2m-1}^1(\mu)\,\mathrm{d}\mu + (2n+1)\int_0^1 P_{2m}^1(\mu)\,P_{2n-1}^1(\mu)\,\mathrm{d}\mu, \tag{B 4}$$

which can be evaluated using equation (B 17) of Hulme (1982).

B.2. *First-order exciting force*

Calculated by integration over the body surface. For waves incident at an angle β to the negative x-axis the components of the exciting force on cylinder j in an array of N cylinders are:

$$f_x^{(1)} \sim -2\pi\mathrm{i}\,\rho g A a h \frac{\tanh kh}{kh}\mu\epsilon\left\{\cos\beta - \epsilon^2 \sum_{\substack{n=1\\ n\neq j}}^{N} \frac{\cos(\beta-2\alpha_{jn})}{\bar{R}_{jn}^2} + O(\epsilon^4)\right\}, \tag{B 5}$$

$$f_y^{(1)} \sim -2\pi\mathrm{i}\,\rho g A a h \frac{\tanh kh}{kh}\mu\epsilon\left\{\sin\beta + \epsilon^2 \sum_{\substack{n=1\\ n\neq j}}^{N} \frac{\sin(\beta-2\alpha_{jn})}{\bar{R}_{jn}^2} + O(\epsilon^4)\right\}. \tag{B 6}$$

For a hemisphere:

$$f_x^{(1)} \sim -\pi \mathrm{i}\, \rho g A a^2 \mu\epsilon \left\{ \cos\beta + \tfrac{1}{2}\mathrm{i}\epsilon^2 \sum_{\substack{n=1\\ n\neq j}}^{N} \frac{\cos\alpha_{jn}}{\bar{R}_{jn}^2} + O(\epsilon^3) \right\}, \tag{B 7}$$

$$f_y^{(1)} \sim -\pi \mathrm{i}\, \rho g A a^2 \mu\epsilon \left\{ \sin\beta - \tfrac{1}{2}\mathrm{i}\epsilon^2 \sum_{\substack{n=1\\ n\neq j}}^{N} \frac{\sin\alpha_{jn}}{\bar{R}_{jn}^2} + O(\epsilon^3) \right\}, \tag{B 8}$$

$$f_z^{(1)} \sim -\rho\pi g A a^2. \tag{B 9}$$

The total force on an array is obtained by summation over j, the $O(\epsilon^2)$-terms within the braces of (B 7) and (B 8) are then identically zero.

B.3. *Radiation damping coefficients*

These may be calculated directly from the first-order exciting forces using a known relation (Mei 1983, pp. 321 and 327). For example, the surge–surge damping coefficient for an array of vertical cylinders is

$$B \sim \pi^2 \rho a^2 h\omega \frac{\tanh kh}{kh} \left(1 + \frac{2kh}{\sinh 2kh}\right)^{-1} (\mu\epsilon)^2 \left\{ N^2 - 2\epsilon^2 N \sum_{j=1}^{N} \sum_{\substack{n=1\\ n\neq j}}^{N} \frac{\cos 2\alpha_{jn}}{\bar{R}_{jn}^2} + O(\epsilon^4) \right\}. \tag{B 10}$$

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